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SUFFICIENT CONDITIONS FOR UNCONSTRAINED OR CONSTRAINED  
MAXIMIZATION OF A QUASI-CONCAVE FUNCTION

BY  
KENNETH ARROW

TECHNICAL REPORT NO. 75  
JUNE 30, 1959

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SUFFICIENT CONDITIONS FOR UNCONSTRAINED OR CONSTRAINED  
MAXIMIZATION OF A QUASI-CONCAVE FUNCTION

by

Kenneth J. Arrow

1. Introduction

For a vector variable  $\underline{x}$ , a necessary condition that  $\bar{x}$  be a maximum of  $f(x)$  if there are no constraints is that,

$$(1) \quad \bar{f}_x = 0.$$

(Bars denote evaluations at  $\bar{x}$ ;  $\underline{x}$  will be a column,  $f_x$  a row vector.)  
For maximization of  $f(x)$  subject to a vector constraint,  $g(x) = 0$ , the usual Lagrange necessary conditions are that, for some row vector  $\bar{y}$ ,

$$(2) \quad \bar{L}_x = 0, \quad \bar{L}_y = 0,$$

where,

$$(3) \quad L(x, y) = f(x) + y g(x).$$

Strictly speaking, the Lagrange conditions are necessary only if the function  $g(x)$  satisfies one or another of the conditions which have been termed, "constraint qualifications," (CQ) (see Bliss [2], p. 210, Kuhn and Tucker [5], or Arrow, Hurwicz, and Uzawa [1]).

Kuhn and Tucker have considered the maximization problems when  $\underline{x}$  is restricted to be non-negative (a vector is non-negative if each component is). The analogue of (1) is,

$$(4) \quad \bar{f}_x \leq 0, \quad \bar{f}_x \bar{x} = 0, \quad \bar{x} \geq 0.$$



That is, (4) is a necessary condition for  $\bar{x}$  to be a maximum of  $f(x)$  for all  $x \geq 0$ . If (CQ) holds, the necessary (Kuhn-Tucker-Lagrange) conditions for a constrained maximum of  $f(x)$  subject to inequality constraints  $g(x) \geq 0$ , where  $x$  again is required to be non-negative, are,

$$\begin{aligned} \text{(KTL)} \quad & \bar{L}_x \leq 0, \bar{L}_x \bar{x} = 0, \bar{x} \geq 0, \\ & \bar{L}_y \geq 0, \bar{y} \bar{L}_y = 0, \bar{y} \geq 0, \end{aligned}$$

where, of course,  $L_y = g(x)$ .

Conditions (4) or (KTL) are sufficient for an unconstrained or constrained maximum, respectively, if the functions  $f(x)$  and  $g(x)$  are concave. We wish to investigate the sufficiency of the conditions under weaker assumptions on the functions, especially on the maximand  $f(x)$ . Specifically, it will be assumed that  $f(x)$  is quasi-concave, by which is meant,

$$(5) \quad \text{for each real number } c, \text{ the set } \{x: f(x) \geq c\} \text{ is concave.}$$

This assumption is of significance in the economic theory of consumer's demand, where  $f(x)$  is the utility function to be maximized subject to a linear constraint (see, for example, Wold [7], Part II).

Under certain global regularity conditions (de Finetti [3], Fenchel [4], Chapter III, Sections 7 and 8), there exists a strictly increasing function  $F(u)$  of the real variable  $u$  such that  $F[f(x)]$  is concave. If  $\bar{x}$  maximizes  $f(x)$  subject to some (possibly null) set of constraints, then  $\bar{x}$  maximizes  $F[f(x)]$  subject to the same constraints and conversely, so that conditions (4) or (KTL) could be applied. However, we can avoid these global regularity conditions (other than differentiability) by a more direct procedure.

Some examples in Section 2 show that conditions (4) or (KTL) are not by themselves sufficient conditions for maxima when  $f(x)$  is quasi-concave. In Section 3, we assume that the vector  $\underline{x}$  can be partitioned into vectors  $x^1, x^2$  such that  $\bar{f}_{x^1} > 0, \bar{f}_{x^2} > 0, \bar{x}^2 = 0$ . It is shown that the function  $f(x^1, 0)$  is maximized at  $\bar{x}^1$  under certain (essentially local) regularity conditions. In Section 4, some theorems which bear on the duality between constrained maxima and minima are presented; from them can be derived a sufficient condition for maximization of a quasi-concave function, namely, that in addition to (4), there must be a corner maximum with respect to at least one variable, that is,  $\bar{f}_{x_i} < 0, \bar{x}_i = 0$  for at least one  $i$ . In Section 5, it is shown that if  $\bar{f}_x \neq 0$  and if certain regularity conditions are satisfied, then (KTL) is sufficient for a constrained maximum. In Section 6 we briefly discuss the relation between constrained maxima and saddle points of the Lagrangian,  $L(x, y)$ .

## 2. Some Examples

Example A. Any monotone function of one variable is clearly quasi-concave. Consider the function,  $f(x) = (x-1)^3$ . At  $\bar{x} = 1, \bar{f}_x = 0$ , so that (4) is satisfied, yet  $\bar{x} = 1$  is certainly not an unconstrained maximum.

Example B. With the same choice for  $f(x)$ , let  $g(x) = 2 - x$ . If  $\bar{x} = 1, \bar{y} = 0$ , (KTL) is satisfied, yet clearly the constrained maximum occurs at  $x = 2$ , not  $x = 1$ .

Example C. These examples can be generalized to any number of dimensions. Let  $F(x)$  be any quasi-concave function,  $\bar{x}$  any point, and define,



$$f(x) = [F(x) - F(\bar{x})]^3.$$

Then  $f(x)$  is quasi-concave and has the same maxima as  $F(x)$ . But  $\bar{F}_x = 0$ , although  $\bar{x}$  was chosen arbitrarily. Similarly, if  $g(x)$  is any vector function for which  $g(\bar{x}) \geq 0$ , (KTL) is satisfied with  $\bar{y} = 0$ , although  $\bar{x}$  certainly need not be the constrained maximum for  $f(x)$  subject to  $g(x) \geq 0$ .

Example D. This is a somewhat more complicated case. Let,

$$(6) \quad f(x) = [x_1 - 1 + [(x_1 - 1)^2 + x_2]^{1/2}]^2.$$

First, we must show that  $f(x)$  is quasi-concave. The condition,  $f(x) \geq c$ , is for  $c \geq 0$ , equivalent to

$$(7) \quad [(x_1 - 1)^2 + x_2]^{1/2} \geq c^{1/2} - (x_1 - 1).$$

(7) is satisfied if  $x_1 - 1 > c^{1/2}$ ; otherwise, we can square both sides of (7) and preserve the inequality. After simplification, we have,

$$(8) \quad 2c^{1/2}(x_1 - 1) + x_2 - c \geq 0.$$

Since (8) is certainly also satisfied (for non-negative  $x_2$ ) when  $x_1 - 1 > c^{1/2}$ , we see immediately that  $\{x: f(x) \geq c\}$  is convex for all non-negative  $c$ , while for negative  $c$ , the set obviously consists of the entire non-negative quadrant.

The function  $f(x)$  is obviously also differentiable when  $(x_1 - 1)^2 + x_2 > 0$ , that is, if either  $x_1 \neq 1$  or  $x_2 > 0$ . To investigate differentiability at  $\bar{x} = (1, 0)$ , note that

$$(9) \quad f(1, x_2) = x_2,$$

$$(10) \quad \begin{aligned} f(x_1, 0) &= 0 && \text{if } x_1 \leq 1, \\ &= 4(x_1 - 1)^2 && \text{if } x_1 > 1. \end{aligned}$$

Then  $f_{x_1}$  and  $f_{x_2}$  are both defined at  $\bar{x}$ , with,

$$(11) \quad \bar{f}_{x_1} = 0, \quad \bar{f}_{x_2} = 1.$$

Consider now the maximization of  $f(x)$  subject to the constraint,  $x_2 \leq 0$ , that is,  $g(x) = -x_2$ . Since  $x_2$  is non-negative, this is equivalent to requiring that  $x_2 = 0$ , and the problem is the unconstrained maximization of  $f(x_1, 0)$ . From (10), there is no maximum, but from (11), it is easily seen that (KTL) is satisfied with  $\bar{y} = 1$ .

### 3. Maximization of $f(x^1, 0)$ .

Example D suggests more intensive study of the general case where  $\underline{x}$  can be partitioned into two subvectors,  $x^1, x^2$ , such that at some point,  $\bar{x}$ ,

$$(12) \quad \bar{f}_{x^1} = 0,$$

while the components of  $\bar{f}_{x^2}$  are all non-zero. We shall see below that if any component of  $\bar{f}_{x^2}$  is negative or if any component of  $\bar{x}^2$  is positive, then the usual maximization conditions are valid. If there are no components to  $x^2$ , so that  $\bar{f}_x = 0$ , then Examples A-C show that nothing much can be expected. In this section, we consider the remaining case where,

$$(13) \quad \bar{f}_{x^2} > 0, \quad \bar{x}^2 = 0.$$

We shall consider again the maximization of  $f(x^1, 0)$ , so that Example D shows that (12-13) imply maximization only under additional regularity conditions. We shall use the following:

(R-1)  $f_{x^2}$  is continuous at  $\bar{x}$ ;

(R-2) for each  $\xi^1$  such that  $\bar{x}^1 + \xi^1 \geq 0$ , if

$$(14) \quad u^* = \inf \{u: f(\bar{x}^1 + u \xi^1, 0) \neq f(\bar{x}^1, 0)\} < 1,$$

then, for some  $a > 1$ ,

$$(15) \quad \limsup_{h \rightarrow 0+} \frac{f[\bar{x}^1 + (u^* + ah) \xi^1, 0] - f(\bar{x}^1 + u^* \xi^1, 0)}{f[\bar{x}^1 + (u^* + h) \xi^1, 0] - f(\bar{x}^1 + u^* \xi^1, 0)} > a.$$

We will first show,

Lemma 1. If  $f(x)$  is differentiable and quasi-concave,  $\bar{f}_{x^1} = 0$ ,  $\bar{f}_{x^2} > 0$ ,  $\bar{x}^2 = 0$ , and if conditions (R-1) and (R-2) hold, then  $\bar{x}^1$  maximizes

$$f(x^1, 0) \text{ for } x^1 \geq 0,$$

and then comment on the regularity assumptions.

Proof of Lemma 1. For any given  $x^1 \geq 0$  and an arbitrary  $\xi^2 > 0$ , define

$$(16) \quad \phi(u, v) = f[\bar{x}^1 + u(x^1 - \bar{x}^1), v \xi^2].$$

It is easy to see that  $\phi(u, v)$  is differentiable and quasi-concave. Also,

$$(17) \quad \phi_v(0, 0) = \bar{f}_{x^2} \xi^2 > 0,$$

$$(18) \quad \phi_u(0, 0) = \bar{f}_{x^1} (x^1 - \bar{x}^1) = 0.$$

From (R-1),

$$(19) \quad \phi_v \text{ is continuous at } (0, 0).$$

We seek to prove that  $f(\bar{x}^1, 0) \geq f(x^1, 0)$  for all  $x^1 \geq 0$ , which is equivalent to  $\phi(0, 0) \geq \phi(1, 0)$ . It therefore suffices to prove that

$\phi(0, 0) \geq \phi(u, 0)$ ,  $0 \leq u \leq 1$ . From (14),

$$(20) \quad \phi(0, 0) = \phi(u, 0), \quad 0 \leq u \leq u^*.$$

Hence, if  $u^* \geq 1$ , the desired result clearly holds. Now suppose

$$(21) \quad u^* < 1.$$

If  $u^* > 0$ , then  $\phi_u(u, 0) = 0$  for  $0 \leq u \leq u^*$ , so that,

$$(22) \quad \phi_u(u^*, 0) = 0.$$

Of course, (22) also holds, from (18), if  $u^* = 0$ .

Suppose that the desired conclusion is false, that is,  $\phi(u, 0) = c_1 > \phi(0, 0)$  for some  $u$ ,  $0 < u \leq 1$ . We shall show that this leads to a contradiction. Let  $\tilde{u}$  be the smallest  $u$  such that  $\phi(\tilde{u}, 0) = c_1$ , so that,

$$(23) \quad \phi(\tilde{u}, 0) = c_1, \quad \phi(u, 0) < c_1, \quad 0 \leq u < \tilde{u}.$$

If  $0 \leq u_1 < u_2 \leq \tilde{u}$ , let  $I = \{u: \phi(u, 0) \geq \phi(u_1, 0)\}$ . By definition,  $u_1 \in I$ ; from (23),  $\tilde{u} \in I$ . But  $I$  is a convex set of real numbers, that is, an interval, so that it must contain  $u_2$  if it contains  $u_1$  and  $u$ . Hence,

$$(24) \quad \phi(u, 0) \text{ is monotone increasing in } [0, \tilde{u}].$$

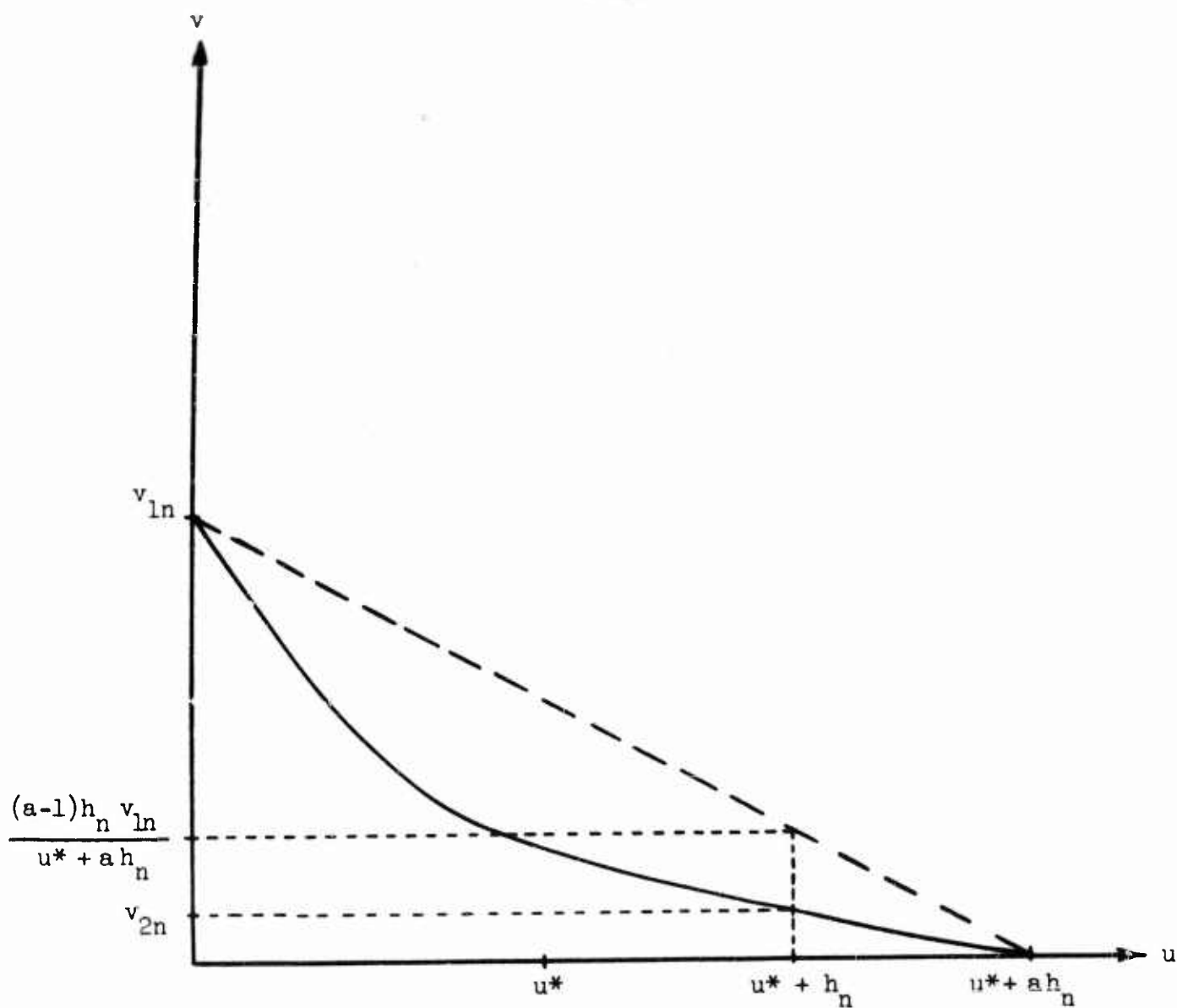
From (24) and (14), we must have,  $\phi(u, 0) > \phi(0, 0)$  for  $u^* < u \leq \tilde{u}$ , so that,

$$(25) \quad \phi(u^* + h) > \phi(0, 0) = \phi(u^*, 0) \quad \text{for } h > 0.$$

and sufficiently small.

The subsequent steps will be clarified by consideration of Figure 1.

Figure 1



The solid curve is a level curve of  $\phi(u, v)$ . First note that from (21) and condition (R-2), we can, with the notation of (16), write (15) in the following form: There exist numbers  $a > 1$  and  $\epsilon > 0$  and a sequence  $\{h_n\}$ , where  $h_n > 0$ ,  $h_n \rightarrow 0$ , such that,

$$(26) \quad \frac{\phi(u^* + ah_n, 0) - \phi(u^*, 0)}{\phi(u^* + h_n, 0) - \phi(u^*, 0)} \geq a + \epsilon, \text{ for } n \text{ sufficiently large.}$$

In view of (17) and (25), we can, for  $n$  sufficiently large, find  $v_{1n}$  so that,

$$(27) \quad \phi(0, v_{1n}) = \phi(u^* + ah_n, 0),$$

where  $v_{1n} \rightarrow 0$ . Form a convex combination of the points  $(0, v_{1n})$  and  $(u^* + ah_n, 0)$  with weights  $(a-1)h_n/(u^* + ah_n)$ ,  $[u^* + h_n]/(u^* + ah_n)$ , respectively. From the quasi-concavity of  $\phi(u, v)$  and (27),

$$(28) \quad \begin{aligned} & \phi[u^* + h_n, (a-1)h_nv_{1n}/(u^* + ah_n)] \\ & \geq \phi(u^* + ah_n, 0) \geq \phi(u^* + h_n, 0), \end{aligned}$$

the last step following from (24). By continuity with respect to  $v$ , we can find  $v_{2n}$  such that,

$$(29) \quad \phi(u^* + h_n, v_{2n}) = \phi(u^* + ah_n, 0),$$

where,

$$(30) \quad 0 \leq v_{2n} \leq (a-1)h_nv_{1n}/(u^* + ah_n).$$

From (26), we deduce,

$$(31) \quad \begin{aligned} \phi(u^* + ah_n, 0) - \phi(u^* + h_n, 0) &= [\phi(u^* + ah_n, 0) - \phi(u^*, 0)] \\ &\quad - [\phi(u^* + h_n, 0) - \phi(u^*, 0)] \\ &\geq [(a+\epsilon-1)/(a+\epsilon)][\phi(u^* + ah_n, 0) - \phi(u^*, 0)] \\ &= [(a+\epsilon-1)/(a+\epsilon)][\phi(0, v_{1n}) - \phi(0, 0)], \end{aligned}$$



the last step following from (27) and (25). From (17),  $\phi(0, v_{1n}) - \phi(0, 0) > 0$  for  $n$  sufficiently large, so that, from (31),

$$\phi(u^* + ah_n, 0) - \phi(u^* + h_n, 0) > 0,$$

and, from (29) and (30),

$$(32) \quad v_{2n} > 0.$$

From (32), we can apply in turn Rolle's Theorem, (29), (31), and (30), to find, for some  $v_{3n}$ ,  $0 \leq v_{3n} \leq v_{2n}$ ,

$$\begin{aligned} (33) \quad \phi_v(u^* + h_n, v_{3n}) &= [\phi(u^* + h_n, v_{2n}) - \phi(u^* + h_n, 0)]/v_{2n} \\ &= [\phi(u^* + ah_n, 0) - \phi(u^* + h_n, 0)]/v_{2n} \\ &\geq [(a + \epsilon - 1)/(a + \epsilon)(a - 1)][(u^* + ah_n)/h_n][\phi(0, v_{1n}) - \phi(0, 0)]/v_{1n}. \end{aligned}$$

Now  $\lim_{n \rightarrow \infty} [\phi(0, v_{1n}) - \phi(0, 0)]/v_{1n} = \phi_v(0, 0) > 0$ . If  $u^* > 0$ , then

$(u^* + ah_n)/h_n \rightarrow \infty$ , so that,

$$\lim_{n \rightarrow \infty} \phi_v(u^* + h_n, v_{3n}) = +\infty.$$

Since  $h_n \rightarrow 0$  and  $v_{3n} \rightarrow 0$ , this contradicts (R-1). If  $u^* = 0$ , then

$(u^* + ah_n)/h_n \equiv a$ . From (33) and (R-1),

$$\phi_v(0, 0) \geq [(a + \epsilon - 1)a/(a + \epsilon)(a - 1)] \phi_v(0, 0).$$

Since the expression in brackets is greater than 1 and  $\phi_v(0, 0) > 0$ , this is impossible. Q.E.D.

Example D fails to satisfy the condition (R-1). For,

$$f_{x^2} = 1 + (x_1 - 1)/[(x_1 - 1)^2 + x_2]^{1/2} \quad \text{if } x \neq (1, 0).$$

Then  $f_{x_2}(1, x_2) = 1$  for  $x_2 > 0$ ,  $f_{x_2}(x_1, 0) = 2$  for  $x_1 > 1$ ,  $0$  for  $x_1 < 1$ .  
Hence  $f_{x_2}$  has different limits as  $(1, 0)$  is approached from different directions.

The significance of condition (R-2) is obscure. It is possible that the other assumptions imply (R-2), but this is an open question. For notational simplicity, let,

$$(34) \quad \psi(h) = f[\bar{x}^1 + (u^* + h) \xi^1, 0] - f(\bar{x}^1 + u^* \xi^1, 0).$$

The desired property is that,

$$(35) \quad \limsup_{h \rightarrow 0+} \frac{\psi(ah)}{\psi(h)} > a, \text{ for some } a > 1.$$

In the notation of the proof of Lemma 1,  $\psi(h) = \phi(u^* + h, 0) - \phi(0, 0)$ , so that, from (22), (34), and (14),

$$(36) \quad \psi(0) = 0, \psi'(0) = 0, \psi(h) \neq 0 \text{ for some } h \text{ in every right-hand neighborhood of } 0.$$

It may be that (36) implies (35). The implication certainly holds if, for some  $n$ ,

$$(37) \quad \psi^{(n)}(0) \neq 0.$$

For in that case, we can evaluate the limit in (35) by L'Hopital's rule. If we use the smallest  $n$  for which (37) holds,

$$(38) \quad \lim_{h \rightarrow 0} \psi(ah)/\psi(h) = a^n \psi^{(n)}(0)/\psi^{(n)}(0) = a^n.$$

From (36) and (37), we must have  $n \geq 2$ ; for any  $a > 1$ ,  $a^n > a$ .

In terms of  $f(x)$ , the condition (37) can be interpreted as follows:  
in every direction  $\xi^1$  such that  $\bar{x}^1 + \xi^1 \geq 0$ , the function  $f(x^1, 0)$   
must be constant for some interval (possibly null) at the end of which  
the  $n^{\text{th}}$  right-hand derivative in that direction must differ from zero  
for some  $n$ .

A still stronger but perhaps more easily applied condition is that  
in every direction  $\xi^1$  such that  $\bar{x}^1 + \xi^1 \geq 0$ , the  $n^{\text{th}}$  directional  
derivative of  $f(x^1, 0)$  is non-zero at  $\bar{x}^1$  for some  $n$ . In this case,  
 $u^*$  must be zero for each  $\xi^1$ , for if  $u^* > 0$ , the directional deriva-  
 tives of all orders would have to be zero, by (14).

Of course, if  $f(x^1, 0)$  is analytic, then (37) and therefore (R-1)  
 must hold, for if  $\psi^{(n)}(0) = 0$  for all  $n$ ,  $\psi(h) = 0$ , which contradicts  
 (36).

#### 4. Duality Between Constrained Maxima and Minima; Sufficient Condition for an Unconstrained Maximum.

In many maximization problems with a single constraint, the value of  
 $\bar{x}$  which maximizes  $f(x)$  subject to  $g(x) \geq 0$  also minimizes  $g(x)$  sub-  
 ject to  $f(x) \geq f(\bar{x})$ . We shall explore conditions involving derivatives  
 of the maximand for this duality when  $f(x)$  is quasi-concave and  $g(x)$  is  
 linear.

Theorem 1. If  $f(x)$  is differentiable and concave,  $\bar{x} \geq 0$ , and  $p$   
 any row vector such that  $p \geq \bar{f}_x$ ,  $p\bar{x} = \bar{f}_x \bar{x}$ , then  $\bar{x}$  minimizes  $px$  sub-  
 ject to  $f(x) \geq f(\bar{x})$ ,  $x \geq 0$ .

Proof: Let  $\underline{x}$  be any point such that,

$$(39) \quad f(x) \geq f(\bar{x}), \quad x \geq 0,$$

and let,

$$(40) \quad \phi(t) = f[(1-t)\bar{x} + tx].$$

From (39) and quasi-concavity,  $\phi(t) \geq f(\bar{x}) = \phi(0)$  for  $0 \leq t \leq 1$ , so that  $\phi'(0) \geq 0$ . From (40), this becomes,

$$(41) \quad \bar{f}_x(x - \bar{x}) \geq 0.$$

But,

$$(42) \quad p(x - \bar{x}) = \bar{f}_x(x - \bar{x}) + (p - \bar{f}_x)x - (p - \bar{f}_x)\bar{x}.$$

Since  $p \geq \bar{f}_x$ ,  $x \geq 0$ , clearly  $(p - \bar{f}_x)x \geq 0$ , while  $(p - \bar{f}_x)\bar{x} = 0$  by hypothesis. From (41) and (42),  $p(x - \bar{x}) \geq 0$ , or  $px \geq p\bar{x}$ .

Theorem 2. Suppose that  $f(x)$  is differentiable and quasi-concave,  $\bar{x} \geq 0$ , and  $\bar{f}_x \neq 0$ , and, if  $\underline{x}$  can be partitioned into  $x^1, x^2$  such that

$$(43) \quad \bar{f}_{x^1} = 0, \quad \bar{f}_{x^2} > 0, \quad \bar{x}^2 = 0,$$

assume also that (R-1) and (R-2) hold. Then  $\bar{x}$  maximizes  $f(x)$  subject to the constraints,  $\bar{f}_x x \leq \bar{f}_x \bar{x}$ ,  $x \geq 0$ .

Proof: If (43) holds, then  $\bar{f}_x x = \bar{f}_{x^1} x^1 + \bar{f}_{x^2} x^2 = \bar{f}_{x^2} x^2$ . In particular,  $\bar{f}_x \bar{x} = \bar{f}_{x^2} \bar{x}^2 = 0$ . The constraint,  $\bar{f}_x x \leq \bar{f}_x \bar{x}$ , is thus equivalent to,  $\bar{f}_{x^2} x^2 \leq 0$ . Since  $\bar{f}_{x^2} > 0$ ,  $x^2 \geq 0$ , the constraint is equivalent to  $x^2 = 0$ . But Lemma 1 insures that  $\bar{x}^1$  maximizes  $f(x^1, 0)$ .

If (43) does not hold, then either  $\bar{f}_{x_1} < 0$  for some  $\underline{1}$  or  $\bar{f}_x \bar{x} > 0$ .

Consider any  $\underline{x}$  such that,

$$(44) \quad f(x) > f(\bar{x}), \quad x \geq 0.$$

We wish to show that  $\bar{f}_x x > \bar{f}_x \bar{x}$ . First suppose that  $\bar{f}_{x_1} < 0$  for some  $\underline{1}$ . Let  $\underline{h}$  be the unit vector in the  $\underline{1}^{\text{th}}$  direction.

$$(45) \quad \bar{f}_x h < 0, \quad h \geq 0.$$

By continuity,

$$(46) \quad f(x + \epsilon h) \geq f(\bar{x}), \quad \epsilon > 0,$$

so that  $x + \epsilon h \geq 0$  from (44) and (45). By Theorem 1, with  $p = \bar{f}_x$ ,  $\bar{f}_x(x + \epsilon h) \geq \bar{f}_x \bar{x}$ , or from (45) and (46),

$$(47) \quad \bar{f}_x x > \bar{f}_x \bar{x},$$

which was to be proved.

Now suppose that  $\bar{f}_x \bar{x} > 0$ . From (44), by continuity, we can choose  $\lambda$  so that

$$(48) \quad f(\lambda x) \geq f(\bar{x}), \quad 0 < \lambda < 1.$$

From (43) and Theorem 1, since  $\lambda x \geq 0$ ,

$$(49) \quad \bar{f}_x(\lambda x) \geq \bar{f}_x \bar{x}.$$

From (48), (49), and the fact that  $\bar{f}_x \bar{x} > 0$ ,

$$\bar{f}_x(\lambda x) > \bar{f}_x(\lambda x) + (\lambda - 1) \bar{f}_x \bar{x} \geq \bar{f}_x \bar{x} + (\lambda - 1) \bar{f}_x \bar{x} = \lambda \bar{f}_x \bar{x},$$

and (47) follows by division through by  $\lambda$ .



Theorem 3. If  $f(x)$  is differentiable and quasi-concave,  $\bar{f}_x \leq 0$ ,  $\bar{f}_x \bar{x} = 0$ , and  $\bar{x} \geq 0$ , then  $\bar{x}$  maximizes  $f(x)$  for  $x \geq 0$ .

By  $\bar{f}_x \leq 0$  is meant  $\bar{f}_x \leq 0$ ,  $\bar{f}_x \neq 0$ . Theorem 3 is the sufficient condition for an unconstrained maximum in non-negative variables.

Proof: Since (43) does not hold, Theorem 2 implies, without further regularity conditions, that  $\bar{x}$  maximizes  $f(x)$  subject to  $\bar{f}_x x \leq \bar{f}_x \bar{x}$ ,  $x \geq 0$ . But from the hypotheses, the condition  $\bar{f}_x x \leq \bar{f}_x \bar{x}$  holds for all  $x \geq 0$ .

#### 5. Sufficient Conditions for a Constrained Maximum.

Theorem 4. Let  $f(x)$  be differentiable and quasi-concave, and suppose that  $\bar{f}_x \neq 0$ , and, if  $\bar{f}_{x_1} = 0$ ,  $\bar{f}_{x_2} > 0$ ,  $\bar{x}^2 = 0$  for some partition of the vector  $\underline{x}$ , then (R-1) and (R-2) hold. Suppose further that for some  $\bar{y}$ , (KTL) holds and that  $\bar{y} g(x)$  is quasi-concave and differentiable, where  $g(x)$  is a vector function. Then  $\bar{x}$  maximizes  $f(x)$  subject to  $g(x) \geq 0$ ,  $x \geq 0$ .

In other words, Theorem 4 states the conditions under which (KTL) is sufficient for a constrained maximum provided  $\bar{y} g(x)$  is quasi-concave.

Proof: From Theorem 2,

(50),  $\bar{x}$  maximizes  $f(x)$  subject to  $\bar{f}_x x \leq \bar{f}_x \bar{x}$ ,  $x \geq 0$ .

From (KTL),  $\bar{L}_x \leq 0$ ,  $\bar{L}_x \bar{x} = 0$ ,  $\bar{x} \geq 0$ ; from (2),

$$-\bar{f}_x \geq \bar{y} \bar{g}_x, -\bar{f}_x \bar{x} = \bar{y} \bar{g}_x \bar{x}.$$

We may apply Theorem 1, with  $f(x)$  replaced by  $\bar{y} g(x)$  and  $\underline{p}$  by  $-\bar{f}_x$ . Then  $\bar{x}$  minimizes  $-\bar{f}_x x$  subject to  $\bar{y} g(x) \geq \bar{y} g(\bar{x})$ ,  $x \geq 0$ . Since, from



(KTL),  $\bar{y} \bar{L}_y = 0$ , which is the same as,  $\bar{y} g(\bar{x}) = 0$ , we have,

$$(51) \quad \bar{x} \text{ maximizes } \bar{f}_x x \text{ subject to } \bar{y} g(x) \geq 0, x \geq 0.$$

Consider any  $x$  such that  $g(x) \geq 0, x \geq 0$ . Then  $\bar{y} g(x) \geq 0$ , since  $\bar{y} \geq 0$  by (KTL). From (51),  $\bar{f}_x x \leq \bar{f}_x \bar{x}$ , and, by (50),  $f(\bar{x}) \geq f(x)$ . Since also  $\bar{L}_y \geq 0$ , which means  $g(\bar{x}) \geq 0$ , the theorem follows.

The condition,  $\bar{y} g(x)$  is quasi-concave, is certainly fulfilled if either  $g(x)$  is one-dimensional and quasi-concave or  $g(x)$  is concave, in the latter case since then  $\bar{y} g(x)$  is concave and hence certainly quasi-concave.

Theorem 5. Let  $f(x)$  and  $g(x)$  be differentiable quasi-concave scalar functions and suppose that, for some  $\bar{x}$  and  $\bar{y}$ , the hypotheses of Theorem 4 hold. Then  $\bar{x}$  maximizes  $f(x)$  subject to  $g(x) \geq 0, x \geq 0$ .

Theorem 6. Let  $f(x)$  be differentiable and quasi-concave, and let  $g(x)$  be a differentiable concave vector function. If, for some  $\bar{x}, \bar{y}$ , the hypotheses of Theorem 4 hold, then  $\bar{x}$  maximizes  $f(x)$  subject to  $g(x) \geq 0, x \geq 0$ .

In the economic application, there is a single linear constraint,

$$(52) \quad g(x) = M - px \geq 0.$$

Clearly, if  $\bar{f}_x = 0$ , then (KTL) is satisfied with  $\bar{y} = 0$ , but there is no necessity that  $\bar{x}$  be a constrained maximum. Let us therefore assume that,

$$(53) \quad \bar{f}_x \neq 0,$$

and, for the moment,

$$(54) \quad M \neq 0.$$

(In the economic context,  $M > 0$ .) Now suppose (KTL) holds. Then,

$$(55) \quad \bar{f}_x \leq \bar{y} p,$$

$$(56) \quad \bar{f}_x \bar{x} = \bar{y} p \bar{x},$$

$$(57) \quad \bar{y} (p\bar{x} - M) = 0,$$

$$(58) \quad \bar{y} \geq 0.$$

If  $\bar{y} = 0$ , then, from (53) and (55),  $\bar{f}_x \leq 0$ , and therefore,  $\bar{f}_{x_i} < 0$  for some  $i$ . If  $\bar{y} > 0$ , then, from (56) and (57),  $\bar{f}_x \bar{x} = \bar{y} M \neq 0$ , from (54). If  $\bar{f}_x \bar{x} < 0$ , then again  $\bar{f}_{x_i} < 0$  for some  $i$ . Otherwise,  $\bar{f}_x \bar{x} > 0$ , so that the hypotheses of Theorem 4 are fulfilled in any case. Hence, if (53) and (54) hold, (KTL) is sufficient for a constrained maximum from either Theorem 5 or Theorem 6.

If  $M = 0$ , additional regularity conditions may be needed. If  $\bar{f}_{x_i} < 0$  for some  $i$ , then indeed Theorem 5 or 6 may be applied without difficulty. Otherwise,  $\bar{f}_x \geq 0$ ; from (55), we must have  $\bar{y} > 0$  and therefore again  $\bar{f}_x \bar{x} = \bar{y} M = 0$ . If all components of  $\bar{f}_x$  are positive, then  $\bar{x} = 0$ . From (55),  $p > 0$ , and from (52),  $0$  is the only point satisfying the constraints, so again there is a maximum at  $\bar{x}$ . Otherwise, we can partition the vector  $\bar{x}$  so that  $\bar{f}_{x_1} = 0$ ,  $\bar{f}_{x_2} > 0$ . Since  $\bar{f}_x \bar{x} = 0$ , we must have  $\bar{x}^2 = 0$ . In this case,  $\bar{x}$  is a maximum if the regularity conditions (R-1) and (R-2) hold but not necessarily otherwise.

# 6. A Remark on Constrained Maxima and Saddle Points

As Kuhn and Tucker have shown ([5], Theorem 2, p. 485; see also Uzawa [6], Theorem 1, p. 33), a sufficient condition that  $\bar{x}$  be a constrained maximum is that, for some  $\bar{y}$ , the pair  $\bar{x}, \bar{y}$  be a saddle point of  $L(x, y)$ . They also show that, if  $f(x)$  and  $g(x)$  are concave, (CQ) holds, and  $\bar{x}$  is a constrained maximum, then, for some  $\bar{y}$ , the pair  $\bar{x}, \bar{y}$  is a saddle point of  $L(x, y)$ .

The question may be raised whether or not this necessity condition generalizes. In the concave differentiable case, one can deduce the saddle point condition by observing that the first line of (KTL) implies that  $\bar{x}$  maximizes  $L(x, \bar{y})$  while the second line implies that  $\bar{y}$  minimizes  $L(\bar{x}, y)$ . The second implication always holds, since  $L(\bar{x}, y)$  is always linear in  $y$ . The first holds whenever  $L(x, \bar{y})$  is a function  $f(x)$  such that (4) is sufficient for  $\bar{x}$  to be a constrained maximum. By Theorem 3, we could remark,

(27) if  $\bar{x}$  is a constrained maximum of  $f(x)$  subject to  $g(x) \geq 0, x \geq 0$ , if  $\bar{x}, \bar{y}$  satisfy (KTL) with  $\bar{L}_x \leq 0$ , and if  $L(x, \bar{y})$  is quasi-concave in  $x$ , then  $\bar{x}, \bar{y}$  is a saddle point of  $L(x, y)$ .

However, the condition that  $L(x, \bar{y})$  be quasi-concave is stringent. The quasi-concavity of  $f(x)$  and  $\bar{y} g(x)$  separately is not sufficient for that of their sum,  $L(x, \bar{y})$ ; even assuming  $g(x)$  concave is not enough. For example, suppose,  $f(x) = x_1^3$ ,  $g(x) = 1 - x_1 - x_2$ . The constrained maximum is attained at  $\bar{x}_1 = 1, \bar{x}_2 = 0$ ; (KTL) is satisfied with  $\bar{y} = 3$ . Then,  $L(x, \bar{y}) = x_1^3 - 3x_1 - 3x_2 + 3$ .

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